



Super-simple group divisible designs with block size 4 and index 5

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ABSTRACT

In this paper, we investigate the existence of a super-simple $(4, 5)$ -GDD of type g^u and show that such a design exists if and only if $u \geq 4$, $g(u-2) \geq 10$, $g(u-1) \equiv 0 \pmod{3}$ and $u(u-1)g^2 \equiv 0 \pmod{12}$.

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1. Introduction

Let K be a set of positive integers. A *group divisible design* (K, λ) -GDD is a triple $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ which satisfies the following properties:

1. \mathcal{X} is a finite set of points,
2. \mathcal{G} is a partition of \mathcal{X} into subsets called groups,
3. \mathcal{B} is a collection of subsets of \mathcal{X} with sizes from K , called blocks, such that every pair of points from distinct groups occurs in exactly λ blocks, and
4. No pair of points belonging to a group occurs in any block.

The *type* of a GDD $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ is the multiset $\{|G| : G \in \mathcal{G}\}$. We usually use the “exponential” notation to describe types: $g_1^{u_1} \cdots g_k^{u_k}$ denotes u_i occurrences of g_i , $1 \leq i \leq k$, in the multiset. If $K = \{k\}$, we write (k, λ) -GDD instead of $(\{k\}, \lambda)$ -GDD. When $\lambda = 1$, it is omitted. Thus a k -GDD is a $(k, 1)$ -GDD.

A *transversal design* $\text{TD}(k, \lambda; n)$ is a (k, λ) -GDD of type n^k . When $\lambda = 1$, we simply write $\text{TD}(k, n)$. A (K, λ) -GDD with group type 1^v is called a *pairwise balanced design*, denoted by (v, K, λ) -PBD. A (k, λ) -GDD with group type 1^v is called a *balanced incomplete block design*, denoted by (v, k, λ) -BIBD.

A design is called *simple* if it contains no repeated blocks. A design $(\mathcal{X}, \mathcal{B})$ is said to be *super-simple* if $|B_1 \cap B_2| \leq 2$ for any two blocks $B_1, B_2 \in \mathcal{B}$ and $B_1 \neq B_2$. When $|B| = 3$ for any $B \in \mathcal{B}$, a super-simple design is just a simple design. When $\lambda = 1$, the designs are necessarily super-simple.

The concept of super-simple designs was introduced by Gronau and Mullin in [16]. The existence of super-simple designs is an interesting extremal problem by itself, but there are also some useful applications, see [1,3,4,19,22].

There are some known results for the existence of super-simple designs, especially for super-simple (v, k, λ) -BIBDs. For $k = 4$, see [2,5,8–10,14,16,18]. For $k = 5$, see [11,12,14,15]. There are also some known results on super-simple $(v, 4, \lambda)$ -RBIBDs, see [13,23]. Super-simple group divisible designs are useful in constructing BIBDs as well as other types of super-simple designs. The existence of a super-simple $(4, \lambda)$ -GDD of type g^u for $\lambda = 2, 3, 4, 6$ has been solved by Cao, Yan and Wei recently.

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Theorem 1.1 ([7,6]). Let $\lambda \in \{2, 3, 4, 6\}$. The necessary conditions for the existence of a super-simple $(4, \lambda)$ -GDD of type g^u are also sufficient except for $(\lambda, g, u) \in \{(3, 2, 5), (4, 3, 5)\}$.

In this paper, we investigate the existence of a super-simple $(4, 5)$ -GDD of type g^u . We shall use direct and recursive constructions to show that the necessary conditions are also sufficient.

Theorem 1.2. There exists a super-simple $(4, 5)$ -GDD of type g^u if and only if $g(u-2) \geq 10$, $g(u-1) \equiv 0 \pmod{3}$, $u \geq 4$ and $u(u-1)g^2 \equiv 0 \pmod{12}$.

2. Direct constructions

In this section, we give some direct constructions of super-simple designs that will be used as master designs or input designs in the next section. These designs have been obtained from difference families after computer-assisted searches. The way to check the super-simplicity is essentially the same as what was used in [5]. The required base block set for some small designs are divided into two parts: P and R , each of the base blocks of P has to be multiplied by m^i with $0 \leq i \leq s-1$ to get s base blocks under certain additive group. For brevity, we just list P , m , s and R in the proof.

Lemma 2.1. There exists a super-simple $(4, 5)$ -GDD of type 2^u for any $u \in \{13, 19\}$.

Proof. Let the point set be $\mathcal{X} = Z_{2u}$ and the group set be $\{\{i, u+i\} : 0 \leq i \leq u-1\}$. The desired super-simple design can be generated by developing all the base blocks modulo $2u$.

$u = 13$: $(m, s) = (5, 3)$
 $P : \{0, 1, 3, 7\} \quad \{0, 1, 9, 11\} \quad \{0, 3, 9, 21\}$
 $R : \{0, 4, 14, 19\}$
 $u = 19$: $(m, s) = (5, 4)$
 $P : \{0, 1, 2, 4\} \quad \{0, 1, 5, 21\} \quad \{0, 2, 6, 31\}$
 $R : \{0, 2, 9, 12\} \quad \{0, 4, 12, 21\} \quad \{0, 7, 15, 23\} \quad \square$

Lemma 2.2. There exists a super-simple $(4, 5)$ -GDD of type 2^u for any $u \in \{10, 34\}$.

Proof. Let the point set be $\mathcal{X} = Z_{2u}$ and the group set be $\{\{i, u+i\} : 0 \leq i \leq u-1\}$. Instead of listing all the required blocks, we only list the base blocks and all the required blocks can be generated from them by $(+2 \pmod{2u})$.

$u = 10$: $\{0, 1, 2, 3\} \quad \{0, 1, 4, 5\} \quad \{0, 1, 6, 8\} \quad \{0, 2, 4, 7\} \quad \{0, 2, 6, 11\} \quad \{0, 3, 4, 8\}$
 $\{0, 3, 6, 12\} \quad \{0, 3, 9, 17\} \quad \{0, 5, 7, 12\} \quad \{0, 5, 9, 13\} \quad \{0, 6, 13, 19\} \quad \{0, 7, 9, 15\}$
 $\{0, 7, 11, 19\} \quad \{0, 8, 17, 19\} \quad \{0, 11, 15, 17\}$
 $u = 34$: $\{0, 23, 39, 51\} \quad \{0, 2, 28, 59\} \quad \{0, 3, 33, 49\} \quad \{0, 5, 29, 47\} \quad \{0, 3, 39, 53\} \quad \{0, 4, 28, 57\}$
 $\{0, 5, 41, 67\} \quad \{0, 5, 8, 49\} \quad \{0, 3, 41, 57\} \quad \{0, 4, 59, 63\} \quad \{0, 7, 9, 55\} \quad \{0, 6, 32, 38\}$
 $\{0, 3, 7, 8\} \quad \{0, 5, 7, 10\} \quad \{0, 4, 10, 60\} \quad \{0, 9, 13, 49\} \quad \{0, 3, 54, 60\} \quad \{0, 9, 11, 25\}$
 $\{0, 10, 25, 27\} \quad \{0, 4, 9, 58\} \quad \{0, 8, 21, 38\} \quad \{0, 10, 54, 65\} \quad \{0, 11, 12, 50\} \quad \{0, 7, 21, 27\}$
 $\{0, 12, 27, 56\} \quad \{0, 15, 39, 43\} \quad \{0, 17, 24, 42\} \quad \{0, 18, 37, 45\} \quad \{0, 19, 26, 45\} \quad \{0, 20, 40, 61\}$
 $\{0, 20, 43, 51\} \quad \{0, 20, 45, 53\} \quad \{0, 20, 47, 55\} \quad \{0, 21, 33, 46\} \quad \{0, 22, 47, 57\} \quad \{0, 22, 51, 59\}$
 $\{0, 23, 35, 53\} \quad \{0, 23, 33, 43\} \quad \{0, 23, 47, 59\} \quad \{0, 1, 29, 51\} \quad \{0, 1, 32, 67\} \quad \{0, 1, 31, 53\}$
 $\{0, 1, 37, 47\} \quad \{0, 1, 39, 65\} \quad \{0, 2, 9, 32\} \quad \{0, 2, 42, 56\} \quad \{0, 2, 13, 18\} \quad \{0, 2, 15, 21\}$
 $\{0, 4, 61, 67\} \quad \{0, 6, 22, 46\} \quad \{0, 13, 31, 37\} \quad \{0, 15, 25, 52\} \quad \{0, 16, 32, 49\} \quad \{0, 17, 45, 65\}$
 $\{0, 35, 49, 61\} \quad \square$

Lemma 2.3. There exists a super-simple $(4, 5)$ -GDD of type 2^u for any $u \in \{16, 22\}$.

Proof. Let the point set be $\mathcal{X} = Z_{2u}$ and the group set be $\{\{i, u+i\} : 0 \leq i \leq u-1\}$. The desired super-simple design can be generated from all the base blocks by $(+2 \pmod{2u})$.

$u = 16$: $(m, s) = (11, 4)$
 $P : \{0, 1, 4, 5\} \quad \{0, 1, 19, 21\} \quad \{0, 1, 7, 9\} \quad \{0, 2, 7, 22\} \quad \{0, 2, 17, 23\}$
 $R : \{0, 2, 8, 10\} \quad \{0, 3, 8, 14\} \quad \{0, 3, 23, 27\} \quad \{0, 4, 17, 24\} \quad \{0, 8, 17, 18\}$
 $u = 22$: $(m, s) = (3, 4)$
 $P : \{0, 1, 2, 3\} \quad \{0, 1, 4, 6\} \quad \{0, 1, 8, 10\} \quad \{0, 4, 8, 25\} \quad \{0, 7, 19, 23\}$
 $\{0, 7, 31, 33\} \quad \{0, 11, 23, 29\}$
 $R : \{0, 2, 16, 32\} \quad \{0, 5, 12, 25\} \quad \{0, 11, 15, 16\} \quad \{0, 14, 29, 37\} \quad \{0, 15, 29, 43\}$
 $\{0, 17, 23, 26\} \quad \{0, 33, 37, 39\} \quad \square$

Lemma 2.4. *There exists a super-simple $(4, 5)$ -GDD of type 3^u for any $u \in \{8, 12, 20\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{3(u-1)} \cup \{a_0, a_1, a_2\}$ and the group set be $\{\{i, u-1+i, 2u-2+i\} : 0 \leq i \leq u-2\} \cup \{a_0, a_1, a_2\}$. The desired super-simple design can be generated from the following base blocks by $(+1 \text{ modulo } 3(u-1))$, where the subscripts on a are evaluated by $(+1 \text{ mod } 3)$.

$u = 8 :$	$\{a_0, 0, 1, 2\}$	$\{a_0, 0, 3, 8\}$	$\{a_0, 0, 4, 6\}$	$\{a_0, 1, 4, 10\}$	$\{a_0, 2, 5, 17\}$	$\{0, 1, 3, 5\}$
	$\{0, 1, 9, 13\}$	$\{0, 1, 12, 17\}$	$\{0, 2, 10, 13\}$	$\{0, 4, 10, 15\}$		
$u = 12 :$	$\{a_0, 0, 1, 2\}$	$\{a_0, 0, 3, 7\}$	$\{a_0, 0, 5, 6\}$	$\{a_0, 1, 4, 10\}$	$\{a_0, 2, 5, 14\}$	$\{0, 6, 15, 23\}$
	$\{0, 2, 4, 8\}$	$\{0, 2, 7, 14\}$	$\{0, 4, 18, 28\}$	$\{0, 5, 13, 20\}$	$\{0, 5, 17, 25\}$	$\{0, 1, 10, 13\}$
	$\{0, 1, 15, 18\}$	$\{0, 2, 12, 19\}$	$\{0, 4, 17, 23\}$			
$u = 20 :$	$\{a_0, 0, 1, 2\}$	$\{a_0, 0, 3, 7\}$	$\{a_0, 0, 5, 6\}$	$\{a_0, 1, 4, 10\}$	$\{a_0, 2, 5, 14\}$	$\{0, 13, 28, 42\}$
	$\{0, 2, 33, 36\}$	$\{0, 4, 22, 26\}$	$\{0, 5, 10, 16\}$	$\{0, 5, 12, 18\}$	$\{0, 7, 14, 22\}$	$\{0, 8, 16, 25\}$
	$\{0, 8, 18, 28\}$	$\{0, 9, 20, 30\}$	$\{0, 10, 21, 35\}$	$\{0, 11, 26, 43\}$	$\{0, 12, 27, 45\}$	$\{0, 12, 28, 41\}$
	$\{0, 1, 21, 55\}$	$\{0, 1, 31, 33\}$	$\{0, 4, 17, 34\}$	$\{0, 2, 8, 32\}$	$\{0, 4, 24, 44\}$	$\{0, 5, 26, 48\}$
	$\{0, 7, 18, 41\}$					

Lemma 2.5. *There exists a super-simple $(4, 5)$ -GDD of type 3^u for any $u \in \{9, 21\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{3u}$ and the group set be $\{\{i, u+i, 2u+i\} : 0 \leq i \leq u-1\}$. The desired design can be generated by developing all the base blocks modulo $3u$.

$u = 9 :$	$(m, s) = (2, 6)$					
$P :$	$\{0, 1, 2, 5\}$					
$R :$	$\{0, 1, 3, 15\}$	$\{0, 1, 7, 13\}$	$\{0, 3, 7, 20\}$	$\{0, 3, 11, 17\}$		
$u = 21 :$	$(m, s) = (2, 4)$					
$P :$	$\{0, 1, 2, 5\}$	$\{0, 1, 6, 12\}$	$\{0, 3, 10, 23\}$	$\{0, 7, 29, 54\}$	$\{0, 9, 26, 44\}$	
$R :$	$\{0, 1, 15, 37\}$	$\{0, 3, 16, 33\}$	$\{0, 3, 31, 34\}$	$\{0, 6, 25, 36\}$	$\{0, 14, 32, 47\}$	

Lemma 2.6. *There exists a super-simple $(4, 5)$ -GDD of type 4^u for any $u \in \{7, 10, 19, 22\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{4u}$ and the group set be $\{\{i, u+i, 2u+i, 3u+i\} : 0 \leq i \leq u-1\}$. The desired super-simple design can be generated by developing all the base blocks modulo $4u$.

$u = 7 :$	$(m, s) = (3, 6)$					
$P :$	$\{0, 1, 3, 11\}$					
$R :$	$\{0, 3, 8, 23\}$	$\{0, 2, 8, 12\}$	$\{0, 1, 12, 17\}$	$\{0, 4, 13, 17\}$		
$u = 10 :$	$(m, s) = (3, 4)$					
$P :$	$\{0, 1, 5, 7\}$	$\{0, 1, 2, 14\}$	$\{0, 1, 18, 33\}$			
$R :$	$\{0, 4, 11, 23\}$	$\{0, 5, 16, 24\}$	$\{0, 7, 24, 32\}$			
$u = 19 :$	$(m, s) = (5, 4)$					
$P :$	$\{0, 1, 2, 4\}$	$\{0, 1, 5, 7\}$	$\{0, 2, 9, 13\}$	$\{0, 4, 8, 17\}$	$\{0, 7, 33, 41\}$	$\{0, 7, 34, 43\}$
$R :$	$\{0, 1, 14, 61\}$	$\{0, 6, 18, 46\}$	$\{0, 6, 40, 68\}$	$\{0, 8, 16, 33\}$	$\{0, 12, 30, 46\}$	$\{0, 14, 43, 60\}$
$u = 22 :$	$(m, s) = (5, 4)$					
$P :$	$\{0, 1, 2, 4\}$	$\{0, 1, 5, 7\}$	$\{0, 2, 6, 10\}$	$\{0, 3, 6, 12\}$	$\{0, 3, 27, 48\}$	$\{0, 6, 29, 61\}$
	$\{0, 8, 33, 49\}$					
$R :$	$\{0, 1, 17, 35\}$	$\{0, 2, 21, 55\}$	$\{0, 7, 18, 52\}$	$\{0, 7, 20, 61\}$	$\{0, 9, 36, 45\}$	$\{0, 17, 34, 53\}$
	$\{0, 18, 39, 57\}$					

Lemma 2.7. *There exists a super-simple $(4, 5)$ -GDD of type 4^{34} .*

Proof. Let the point set be $\mathcal{X} = Z_{136}$ and the group set be $\{\{i, 34+i, 68+i, 102+i\} : 0 \leq i \leq 33\}$. Instead of listing all the required blocks, we only list the base blocks and all the required blocks can be generated from them by $(+1 \text{ mod } 136)$.

$\{0, 32, 65, 98\}$	$\{0, 4, 60, 98\}$	$\{0, 6, 51, 92\}$	$\{0, 7, 66, 74\}$	$\{0, 5, 52, 59\}$	$\{0, 3, 65, 75\}$
$\{0, 3, 12, 64\}$	$\{0, 5, 56, 62\}$	$\{0, 4, 71, 79\}$	$\{0, 8, 74, 128\}$	$\{0, 6, 47, 128\}$	$\{0, 7, 62, 83\}$
$\{0, 7, 35, 121\}$	$\{0, 3, 9, 13\}$	$\{0, 5, 14, 125\}$	$\{0, 9, 27, 66\}$	$\{0, 9, 26, 109\}$	$\{0, 10, 99, 124\}$
$\{0, 10, 106, 121\}$	$\{0, 11, 103, 116\}$	$\{0, 7, 26, 79\}$	$\{0, 3, 16, 121\}$	$\{0, 11, 48, 66\}$	$\{0, 11, 46, 124\}$
$\{0, 12, 32, 51\}$	$\{0, 4, 26, 120\}$	$\{0, 12, 29, 43\}$	$\{0, 10, 55, 122\}$	$\{0, 17, 90, 113\}$	$\{0, 19, 38, 101\}$
$\{0, 13, 61, 113\}$	$\{0, 13, 37, 100\}$	$\{0, 15, 56, 109\}$	$\{0, 11, 65, 122\}$	$\{0, 17, 48, 76\}$	$\{0, 20, 85, 107\}$
$\{0, 19, 48, 112\}$	$\{0, 18, 76, 104\}$	$\{0, 16, 77, 103\}$	$\{0, 1, 31, 81\}$	$\{0, 2, 25, 103\}$	$\{0, 21, 45, 114\}$
$\{0, 2, 89, 109\}$	$\{0, 1, 41, 45\}$	$\{0, 5, 69, 96\}$	$\{0, 5, 48, 97\}$	$\{0, 3, 86, 104\}$	$\{0, 1, 29, 135\}$
$\{0, 17, 63, 106\}$	$\{0, 1, 37, 97\}$	$\{0, 2, 26, 108\}$	$\{0, 2, 41, 92\}$	$\{0, 6, 52, 84\}$	$\{0, 21, 42, 80\}$
$\{0, 21, 58, 94\}$					

Lemma 2.8. *There exists a super-simple $(4, 5)$ -GDD of type 6^u for any $u \in \{5, 9, 11\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{6u}$ and the group set be $\{\{i, u+i, \dots, 5u+i\} : 0 \leq i \leq u-1\}$. The desired super-simple design can be generated from all the base blocks by $(+1 \bmod 6u)$.

$u = 5 :$ $(m, s) = (13, 3)$
 $P : \{0, 1, 2, 4\}$ $\{0, 1, 9, 27\}$
 $R : \{0, 1, 7, 13\}$ $\{0, 1, 12, 24\}$ $\{0, 2, 6, 13\}$ $\{0, 2, 9, 16\}$
 $u = 9 :$ $(m, s) = (7, 4)$
 $P : \{0, 1, 2, 4\}$ $\{0, 1, 5, 7\}$ $\{0, 2, 15, 32\}$ $\{0, 3, 16, 28\}$
 $R : \{0, 1, 8, 30\}$ $\{0, 4, 10, 41\}$ $\{0, 4, 12, 33\}$ $\{0, 8, 23, 34\}$
 $u = 11 :$ $(m, s) = (5, 3)$
 $P : \{0, 1, 2, 4\}$ $\{0, 1, 5, 7\}$ $\{0, 1, 8, 10\}$ $\{0, 3, 7, 21\}$
 $\{0, 3, 37, 54\}$ $\{0, 4, 42, 49\}$ $\{0, 6, 30, 49\}$
 $R : \{0, 8, 21, 38\}$ $\{0, 8, 26, 39\}$ $\{0, 12, 26, 40\}$ $\{0, 13, 27, 51\}$ \square

Lemma 2.9. *There exists a super-simple $(4, 5)$ -GDD of type 6^u for any $u \in \{8, 14\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{6u}$ and the group set be $\{\{i, u+i, \dots, 5u+i\} : 0 \leq i \leq u-1\}$. Instead of listing all the required blocks, we only list the base blocks and all the required blocks can be generated from them by $(+2 \bmod 6u)$.

$u = 8 :$ $\{0, 17, 21, 30\}$ $\{0, 3, 17, 23\}$ $\{0, 5, 23, 25\}$ $\{0, 1, 27, 45\}$ $\{0, 4, 21, 33\}$
 $\{0, 10, 35, 47\}$ $\{0, 7, 18, 28\}$ $\{0, 2, 31, 43\}$ $\{0, 3, 6, 36\}$ $\{0, 5, 27, 31\}$
 $\{0, 6, 13, 18\}$ $\{0, 9, 10, 19\}$ $\{0, 10, 21, 22\}$ $\{0, 11, 13, 20\}$ $\{0, 11, 14, 26\}$
 $\{0, 12, 25, 29\}$ $\{0, 13, 19, 22\}$ $\{0, 13, 23, 28\}$ $\{0, 14, 29, 34\}$ $\{0, 14, 31, 41\}$
 $\{0, 14, 33, 39\}$ $\{0, 15, 27, 41\}$ $\{0, 15, 22, 43\}$ $\{0, 15, 25, 35\}$ $\{0, 1, 19, 37\}$
 $\{0, 1, 28, 31\}$ $\{0, 1, 2, 46\}$ $\{0, 1, 3, 7\}$ $\{0, 2, 6, 44\}$ $\{0, 2, 9, 35\}$
 $\{0, 4, 9, 11\}$ $\{0, 5, 19, 47\}$ $\{0, 5, 30, 39\}$ $\{0, 6, 35, 37\}$ $\{0, 11, 22, 45\}$
 $u = 14 :$ $\{0, 6, 26, 33\}$ $\{0, 18, 41, 65\}$ $\{0, 2, 53, 63\}$ $\{0, 2, 12, 65\}$ $\{0, 4, 31, 51\}$
 $\{0, 26, 67, 75\}$ $\{0, 1, 6, 41\}$ $\{0, 4, 35, 59\}$ $\{0, 5, 43, 55\}$ $\{0, 8, 39, 52\}$
 $\{0, 4, 55, 71\}$ $\{0, 3, 5, 35\}$ $\{0, 5, 9, 57\}$ $\{0, 6, 13, 59\}$ $\{0, 7, 61, 81\}$
 $\{0, 3, 7, 38\}$ $\{0, 8, 16, 26\}$ $\{0, 6, 63, 78\}$ $\{0, 5, 51, 59\}$ $\{0, 9, 11, 33\}$
 $\{0, 8, 63, 66\}$ $\{0, 7, 27, 67\}$ $\{0, 9, 15, 77\}$ $\{0, 10, 29, 34\}$ $\{0, 3, 37, 83\}$
 $\{0, 13, 39, 71\}$ $\{0, 11, 23, 45\}$ $\{0, 10, 46, 83\}$ $\{0, 12, 25, 27\}$ $\{0, 11, 17, 77\}$
 $\{0, 12, 43, 48\}$ $\{0, 25, 79, 83\}$ $\{0, 13, 21, 79\}$ $\{0, 15, 17, 50\}$ $\{0, 13, 22, 68\}$
 $\{0, 11, 12, 62\}$ $\{0, 10, 21, 57\}$ $\{0, 18, 52, 75\}$ $\{0, 15, 55, 61\}$ $\{0, 19, 25, 31\}$
 $\{0, 9, 20, 52\}$ $\{0, 16, 33, 52\}$ $\{0, 19, 22, 62\}$ $\{0, 16, 43, 54\}$ $\{0, 21, 30, 83\}$
 $\{0, 16, 40, 60\}$ $\{0, 21, 46, 69\}$ $\{0, 1, 33, 40\}$ $\{0, 19, 23, 35\}$ $\{0, 15, 24, 54\}$
 $\{0, 17, 29, 49\}$ $\{0, 17, 47, 81\}$ $\{0, 5, 20, 50\}$ $\{0, 2, 69, 71\}$ $\{0, 1, 9, 19\}$
 $\{0, 1, 37, 45\}$ $\{0, 1, 4, 63\}$ $\{0, 2, 37, 47\}$ $\{0, 2, 39, 66\}$ $\{0, 3, 22, 48\}$
 $\{0, 3, 61, 71\}$ $\{0, 4, 29, 77\}$ $\{0, 8, 29, 73\}$ $\{0, 24, 49, 67\}$ $\{0, 43, 61, 77\}$ \square

Lemma 2.10. *There exists a super-simple $(4, 5)$ -GDD of type 6^6 .*

Proof. Let the point set be $\mathcal{X} = Z_{36}$ and the group set be $\{\{i, 6+i, \dots, 30+i\} : 0 \leq i \leq 5\}$. Let $(m, s) = (5, 3)$. The desired super-simple design can be generated from all the base blocks by $(+2 \bmod 36)$.

$P : \{0, 1, 2, 3\}$ $\{0, 1, 4, 5\}$ $\{0, 2, 21, 22\}$ $\{0, 2, 19, 35\}$ $\{0, 2, 9, 17\}$ $\{0, 3, 17, 19\}$
 $R : \{0, 1, 23, 27\}$ $\{0, 4, 13, 20\}$ $\{0, 4, 23, 32\}$ $\{0, 7, 9, 28\}$ $\{0, 8, 21, 35\}$ $\{0, 11, 16, 27\}$
 $\{0, 19, 27, 29\}$ \square

Lemma 2.11. *There exists a super-simple $(4, 5)$ -GDD of type 9^5 .*

Proof. Let the point set be $\mathcal{X} = Z_{45}$ and the group set be $\{\{i, 5+i, \dots, 40+i\} : 0 \leq i \leq 4\}$. Let $(m, s) = (2, 6)$. The desired super-simple design can be generated by developing all the base blocks modulo 45.

$P : \{0, 1, 2, 8\}$ $\{0, 2, 9, 26\}$
 $R : \{0, 3, 9, 21\}$ $\{0, 3, 22, 36\}$ $\{0, 6, 17, 24\}$ \square

Lemma 2.12. *There exists a super-simple $(4, 5)$ -GDD of type 12^u for any $u \in \{5, 6, 11\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{12u}$ and the group set be $\{\{i, u+i, \dots, 11u+i\} : 0 \leq i \leq u-1\}$. The desired super-simple design can be generated from all the base blocks by $(+1 \bmod 12u)$.

$$\begin{aligned}
 u = 5 : & \quad (m, s) = (7, 2) \\
 P : & \quad \{0, 1, 23, 39\} \quad \{0, 1, 13, 17\} \quad \{0, 2, 21, 53\} \quad \{0, 2, 18, 49\} \quad \{0, 2, 39, 48\} \quad \{0, 6, 53, 57\} \\
 & \quad \{0, 8, 14, 56\} \quad \{0, 9, 26, 38\} \quad \{0, 11, 32, 59\} \quad \{0, 23, 41, 49\} \\
 R : & \quad \emptyset \\
 u = 6 : & \quad (m, s) = (5, 4) \\
 P : & \quad \{0, 1, 2, 4\} \quad \{0, 1, 8, 39\} \quad \{0, 1, 59, 68\} \quad \{0, 8, 21, 35\} \\
 R : & \quad \{0, 9, 26, 49\} \quad \{0, 14, 43, 69\} \quad \{0, 1, 16, 44\} \quad \{0, 4, 26, 69\} \quad \{0, 8, 46, 57\} \quad \{0, 14, 27, 34\} \\
 & \quad \{0, 16, 56, 59\} \quad \{0, 20, 40, 57\} \quad \{0, 10, 21, 49\} \\
 u = 11 : & \quad (m, s) = (25, 5) \\
 P : & \quad \{0, 1, 2, 58\} \quad \{0, 1, 5, 6\} \quad \{0, 1, 7, 9\} \quad \{0, 2, 5, 8\} \quad \{0, 2, 6, 9\} \quad \{0, 3, 9, 94\} \\
 & \quad \{0, 4, 24, 60\} \quad \{0, 9, 26, 38\} \quad \{0, 9, 29, 76\} \quad \{0, 10, 23, 46\} \\
 R : & \quad \emptyset \quad \square
 \end{aligned}$$

Lemma 2.13. *There exists a $(\{4, 5\}, 1)$ -GDD of type 2^u for any $u \in \{15, 18\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{2u}$ and the group set be $\{\{i, u+i\} : 0 \leq i \leq u-1\}$. Instead of listing all the required blocks, we only list the base blocks and all the required blocks can be generated from them by $(+2 \bmod 2u)$.

$$\begin{aligned}
 u = 15 : & \quad \{0, 1, 2, 6, 14\} \quad \{0, 3, 7, 10\} \quad \{0, 5, 13, 19\} \quad \{0, 9, 11, 21\} \\
 u = 18 : & \quad \{0, 1, 2, 6, 14\} \quad \{0, 3, 7, 17\} \quad \{0, 5, 21, 33\} \quad \{0, 9, 15, 26\} \quad \{0, 11, 13, 20\} \quad \square
 \end{aligned}$$

Lemma 2.14. *There exists a $(\{4, 5\}, 1)$ -GDD of type 2^{23} .*

Proof. Let the point set be $\mathcal{X} = Z_{46}$ and the group set be $\{\{i, 23+i\} : 0 \leq i \leq 22\}$. All the required blocks can be generated from three base blocks $\{0, 5, 11, 25, 33\}$, $\{0, 1, 4, 16\}$ and $\{0, 2, 9, 19\}$ by $(+1 \bmod 46)$. \square

Lemma 2.15. *There exists a $(\{4, 5\}, 5)$ -GDD of type 2^u for any $u \in \{12, 18\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{2u}$ and the group set be $\{\{i, u+i\} : 0 \leq i \leq u-1\}$. Instead of listing all the required blocks, we only list the base blocks and all the required blocks can be generated from them by $(+2 \bmod 2u)$.

$$\begin{aligned}
 u = 12 : & \quad \{0, 1, 2, 3, 6\} \quad \{0, 1, 5, 7, 8\} \quad \{0, 1, 9, 10\} \quad \{0, 1, 11, 14\} \quad \{0, 2, 4, 7\} \\
 & \quad \{0, 2, 8, 10\} \quad \{0, 3, 4, 9\} \quad \{0, 3, 5, 10\} \quad \{0, 3, 7, 17\} \quad \{0, 4, 8, 19\} \\
 & \quad \{0, 5, 13, 18\} \quad \{0, 6, 13, 16\} \quad \{0, 6, 15, 17\} \quad \{0, 7, 13, 23\} \quad \{0, 9, 13, 17\} \\
 & \quad \{0, 9, 15, 19\} \quad \{0, 13, 15, 21\} \\
 u = 18 : & \quad \{0, 1, 2, 3, 6\} \quad \{0, 1, 5, 7, 8\} \quad \{0, 4, 16, 25\} \quad \{0, 9, 17, 23\} \quad \{0, 4, 8, 30\} \\
 & \quad \{0, 3, 13, 26\} \quad \{0, 2, 7, 19\} \quad \{0, 5, 6, 16\} \quad \{0, 3, 5, 29\} \quad \{0, 3, 19, 35\} \\
 & \quad \{0, 4, 21, 28\} \quad \{0, 5, 14, 20\} \quad \{0, 7, 12, 27\} \quad \{0, 6, 13, 19\} \quad \{0, 3, 11, 20\} \\
 & \quad \{0, 7, 11, 16\} \quad \{0, 8, 23, 25\} \quad \{0, 2, 10, 11\} \quad \{0, 1, 21, 29\} \quad \{0, 2, 14, 24\} \\
 & \quad \{0, 2, 17, 31\} \quad \{0, 9, 21, 35\} \quad \{0, 9, 25, 31\} \quad \{0, 11, 14, 33\} \quad \{0, 11, 15, 23\} \\
 & \quad \{0, 13, 15, 25\} \quad \{0, 23, 27, 33\} \quad \square
 \end{aligned}$$

Lemma 2.16. *There exists a $(\{4, 5\}, 5)$ -GDD of type 2^u for any $u \in \{15, 23\}$.*

Proof. Let the point set be $\mathcal{X} = Z_{2u}$ and the group set be $\{\{i, u+i\} : 0 \leq i \leq u-1\}$. Instead of listing all the required blocks, we only list the base blocks and all the required blocks can be generated from them by $(+1 \bmod 2u)$.

$$\begin{aligned}
 u = 15 : & \quad \{0, 1, 2, 4, 7\} \quad \{0, 1, 5, 9\} \quad \{0, 1, 8, 10\} \quad \{0, 1, 11, 13\} \quad \{0, 2, 6, 11\} \\
 & \quad \{0, 3, 11, 19\} \quad \{0, 3, 13, 20\} \quad \{0, 3, 14, 21\} \quad \{0, 4, 13, 18\} \quad \{0, 5, 12, 18\} \\
 & \quad \{0, 6, 14, 20\} \\
 u = 23 : & \quad \{0, 1, 2, 4, 7\} \quad \{0, 1, 5, 9, 10\} \quad \{0, 10, 24, 35\} \quad \{0, 7, 16, 34\} \quad \{0, 6, 18, 26\} \\
 & \quad \{0, 3, 8, 14\} \quad \{0, 3, 9, 16\} \quad \{0, 5, 15, 33\} \quad \{0, 7, 21, 36\} \quad \{0, 8, 18, 29\} \\
 & \quad \{0, 1, 17, 20\} \quad \{0, 2, 16, 29\} \quad \{0, 2, 22, 35\} \quad \{0, 2, 17, 21\} \quad \{0, 4, 12, 26\} \\
 & \quad \{0, 6, 21, 30\} \quad \{0, 7, 19, 31\} \quad \square
 \end{aligned}$$

Lemma 2.17. *There exists a $(\{4, 5\}, 5)$ -GDD of type 4^{14} .*

Proof. Let the point set be $\mathcal{X} = Z_{56}$ and the group set be $\{\{i, 14 + i, 28 + i, 42 + i\} : 0 \leq i \leq 13\}$. Instead of listing all the required blocks, we only list the base blocks and all the required blocks can be generated from them by $(+1 \bmod 56)$.

$\{0, 1, 2, 4, 7\}$	$\{0, 11, 23, 41\}$	$\{0, 5, 22, 35\}$	$\{0, 7, 30, 38\}$	$\{0, 2, 36, 38\}$	$\{0, 3, 7, 13\}$
$\{0, 3, 8, 11\}$	$\{0, 5, 11, 17\}$	$\{0, 7, 15, 22\}$	$\{0, 10, 22, 37\}$	$\{0, 9, 18, 38\}$	$\{0, 10, 26, 39\}$
$\{0, 1, 9, 25\}$	$\{0, 4, 19, 40\}$	$\{0, 4, 21, 44\}$	$\{0, 1, 13, 24\}$	$\{0, 1, 10, 35\}$	$\{0, 2, 27, 37\}$
$\{0, 4, 20, 31\}$	$\{0, 5, 24, 37\}$	$\{0, 6, 23, 32\}$	\square		

3. Main results

In this section we complete our proof of the sufficiency of the necessary conditions. For our recursive constructions, we shall use the following standard recursive constructions, the proofs of which can be found in [5,9].

Construction 3.1 (Weighting). Let $(\mathcal{X}, \mathcal{G}, \mathcal{B})$ be a super-simple GDD with index λ_1 , and let $\omega : X \rightarrow Z^+ \cup \{0\}$ be a weighting function on X , where Z^+ is the set of positive integers. Suppose that for each block $B \in \mathcal{B}$, there exists a super-simple (k, λ_2) -GDD of type $\{\omega(x) : x \in B\}$. Then there exists a super-simple $(k, \lambda_1 \lambda_2)$ -GDD of type $\{\sum_{x \in G_i} \omega(x) : G_i \in \mathcal{G}\}$.

Construction 3.2 (Breaking up Groups). If there exists a super-simple (K, λ) -GDD of type $(sh_1)^{u_1} \cdots (sh_t)^{u_t}$ and a super-simple (K, λ) -GDD of type $s^{h_i+\eta}$ for each i , $1 \leq i \leq t$, then there exists a super-simple (K, λ) -GDD of type $s^{\sum_{i=1}^t h_i u_i + \eta}$, where $\eta = 0$ or 1 .

To prove our main results, we also need the following known results.

Theorem 3.3 ([5]). There exists a super-simple $(v, 4, 5)$ -BIBD if and only if $v \equiv 1, 4 \pmod{12}$ and $v \geq 13$.

Theorem 3.4 ([12]). A super-simple $(v, 5, 5)$ -BIBD exists if and only if $v \equiv 1 \pmod{4}$ and $v \geq 17$, except possibly when $v = 21$.

Lemma 3.5 ([20,24]).

1. There exists a $(v, \{4, 5, 6\}, 1)$ -PBD for all $v \geq 13$ and $v \notin \{14, 15, 18, 19, 23\}$.
2. A 4-GDD of type g^u exists if and only if $u \geq 4$, $(u-1)g \equiv 0 \pmod{3}$ and $u(u-1)g^2 \equiv 0 \pmod{12}$ except $(g, u) \in \{(2, 4), (6, 4)\}$.

Lemma 3.6 ([17]). A super-simple TD $(4, \lambda; v)$ exists if and only if $\lambda \leq v$ and (λ, v) is neither $(1, 2)$ nor $(1, 6)$.

Lemma 3.7. Let $m \geq 12$, $m \notin \{13, 14, 17, 18, 22\}$. If there exists a super-simple $(4, 5)$ -GDD of type ω^h for any $h \in \{4, 5, 6\}$, then there exists a super-simple $(4, 5)$ -GDD of type $(3\omega)^a(4\omega)^b(5\omega)^c$ for certain $a, b, c \geq 0$.

Proof. By Lemma 3.5, we have an $(m+1, \{4, 5, 6\}, 1)$ -PBD based on \mathcal{X} . Deleting a point from \mathcal{X} , then we get a $\{4, 5, 6\}$ -GDD of type $3^a 4^b 5^c$ for certain $a, b, c \geq 0$. Applying Construction 3.1 with weight $\omega \in N$ and a super-simple $(4, 5)$ -GDD of type ω^h for any $h \in \{4, 5, 6\}$, we get a super-simple $(4, 5)$ -GDD of type $(3\omega)^a(4\omega)^b(5\omega)^c$. \square

Now we are ready to prove Theorem 1.2. It is easy to see that the necessary conditions for a super-simple $(4, 5)$ -GDD of type g^u can be distinguished into four cases: (1) $g \equiv 1, 5 \pmod{6}$, $u \equiv 1, 4 \pmod{12}$ and $g(u-2) \geq 10$; (2) $g \equiv 3 \pmod{6}$, $u \equiv 0, 1 \pmod{4}$ and $g(u-2) \geq 10$; (3) $g \equiv 0 \pmod{6}$, $u \geq 4$ and $g(u-2) \geq 10$; (4) $g \equiv 2, 4 \pmod{6}$, $u \equiv 1 \pmod{3}$ and $g(u-2) \geq 10$.

3.1. Case 1

Theorem 3.8. There exists a super-simple $(4, 5)$ -GDD of type g^u for $g \equiv 1, 5 \pmod{6}$, $u \equiv 1, 4 \pmod{12}$ and $g(u-2) \geq 10$.

Proof. For $u = 4$, the existence of a super-simple $(4, 5)$ -GDD of type g^u comes from Lemma 3.6. For any $u \geq 13$, we start from a super-simple $(u, 4, 5)$ -BIBD which exists by Theorem 3.3. Applying Construction 3.1 with weight g , we obtain a super-simple $(4, 5)$ -GDD of type g^u , where the input 4-GDD of type g^4 comes from Lemma 3.5. \square

3.2. Case 2

In this subsection, we settle the case $g \equiv 3 \pmod{6}$, $u \equiv 0, 1 \pmod{4}$ and $g(u-2) \geq 10$.

Lemma 3.9. There exists a super-simple $(4, 5)$ -GDD of type 3^u for $u \equiv 1 \pmod{4}$ and $u \geq 9$.

Proof. For $u \in \{9, 21\}$, the associated super-simple designs are given in Lemma 2.5.

For $u = 13$, take a super-simple $(13, 4, 5)$ -BIBD from Theorem 3.3. Applying Construction 3.1 with a 4-GDD of type 3^4 , we obtain a super-simple $(4, 5)$ -GDD of type 3^{13} .

For other values of u , we start from a super-simple $(u, 5, 5)$ -BIBD which exists by Theorem 3.4. Applying Construction 3.1 with a 4-GDD of type 3^5 coming from Lemma 3.5, we obtain a super-simple $(4, 5)$ -GDD of type 3^u . \square

Lemma 3.10. *There exists a super-simple $(4, 5)$ -GDD of type 3^u for $u \equiv 0 \pmod{4}$ and $u \geq 8$.*

Proof. For $u \in \{8, 12, 20\}$, the associated super-simple designs are given in Lemma 2.4.

For other values of u , we start from a super-simple $(u + 1, 5, 5)$ -BIBD which exists by Theorem 3.4. Delete a point from the point set to obtain a super-simple $(u, \{4, 5\}, 5)$ -PBD. Applying Construction 3.1 with a 4-GDD of type 3^4 or 3^5 , we obtain a super-simple $(4, 5)$ -GDD of type 3^u . \square

Lemma 3.11. *There exists a super-simple $(4, 5)$ -GDD of type g^5 for $g \equiv 3 \pmod{6}$.*

Proof. Let $g = 6m + 3$ and $m \geq 1$. For $m = 1$, there exists a super-simple $(4, 5)$ -GDD of type 9^5 by Lemma 2.11.

For $m \geq 2$, we have a 4-GDD of type 3^5 by Lemma 3.5. Applying Construction 3.1 with a super-simple $(4, 5)$ -GDD of type $(2m + 1)^4$ coming from Lemma 3.6, we get a super-simple $(4, 5)$ -GDD of type g^5 . \square

Lemma 3.12. *There exists a super-simple $(4, 5)$ -GDD of type g^u for $g \equiv 3 \pmod{6}$, $g \geq 9$ and $u \equiv 0, 1 \pmod{4}$, $u \geq 8$.*

Proof. Let $g = 6m + 3$, $m \geq 1$. There exists a super-simple $(4, 5)$ -GDD of type 3^u by Lemmas 3.9 and 3.10. Applying Construction 3.1 with a 4-GDD of type $(2m + 1)^4$ coming from Lemma 3.5, we get a super-simple $(4, 5)$ -GDD of type g^u . \square

Combining Lemmas 3.6, 3.9 and 3.12, we have the following theorem.

Theorem 3.13. *There exists a super-simple $(4, 5)$ -GDD of type g^u for $g \equiv 3 \pmod{6}$, $u \equiv 0, 1 \pmod{4}$ and $g(u - 2) \geq 10$.*

3.3. Case 3

Now, we deal with the case $g \equiv 0 \pmod{6}$, $u \geq 4$ and $g(u - 2) \geq 10$. First, we need the notation of H design which can be found in [21]. An $H(v, g, k, t)$ design is a triple $(X, \mathcal{G}, \mathcal{B})$, where X is a set of points whose cardinality is vg , and $\mathcal{G} = \{G_1, \dots, G_v\}$ is a partition of X into v sets of cardinality g ; the members of \mathcal{G} are called *groups*. A *transverse* of \mathcal{G} is a subset of X that meets each group in at most one point. The set \mathcal{B} contains k -element transverses of \mathcal{G} , called *blocks*, with the property that each t -element transverse of \mathcal{G} is contained in precisely one block. From the definition, we know that an $H(v, g, 4, 3)$ design is a super-simple $(4, g(v - 2)/2)$ -GDD of type g^v .

Lemma 3.14 ([21]). *There exists an $H(7, 2, 4, 3)$ design, which is also a super-simple $(4, 5)$ -GDD of type 2^7 .*

Lemma 3.15. *There exists a super-simple $(4, 5)$ -GDD of type 6^u for any $u \in \{7, 10, 19\}$.*

Proof. There exists a super-simple $(4, 5)$ -GDD of type 2^u by Lemmas 3.14, 2.2 and 2.1. Applying Construction 3.1 with a 4-GDD of type 3^4 , we get a super-simple $(4, 5)$ -GDD of type 6^u . \square

Lemma 3.16. *There exists a super-simple $(4, 5)$ -GDD of type 6^u for any $u \in \{12, 15, 18, 23\}$.*

Proof. There exists a super-simple $(\{4, 5\}, 5)$ -GDD of type 2^u by Lemmas 2.15 and 2.16. Applying Construction 3.1 with a $(4, 1)$ -GDD of type 3^4 or 3^5 , we get a super-simple $(4, 5)$ -GDD of type 6^u . \square

Lemma 3.17. *There exists a super-simple $(4, 5)$ -GDD of type 6^u for any $u \geq 4$.*

Proof. For $u \in [4, 12] \cup \{14, 15, 18, 19, 23\}$, the associated super-simple designs are given in Lemmas 3.6, 2.8–2.10, 3.15 and 3.16.

For other values of u , take a $(u, \{4, 5, 6\}, 1)$ -PBD from Lemma 3.5. Then apply Construction 3.1 with weight 6 to get a super-simple $(4, 5)$ -GDD of type 6^u . \square

Lemma 3.18. *There exists a super-simple $(4, 5)$ -GDD of type 12^u for any $u \in \{7, 8, 9, 10, 12, 14, 15, 18, 19, 23\}$.*

Proof. For $u \in \{8, 9, 12\}$, there exists a super-simple $(4, 5)$ -GDD of type 3^u by Theorem 3.13. Applying Construction 3.1 with a 4-GDD of type 4^4 coming from Lemma 3.5, we get a super-simple $(4, 5)$ -GDD of type 12^u .

For $u \in \{7, 10, 19\}$, we have a super-simple $(4, 5)$ -GDD of type 4^u by Lemma 2.6. Applying Construction 3.1 with a 4-GDD of type 3^4 , we get a super-simple $(4, 5)$ -GDD of type 12^u .

For $u = 14$, start from a super-simple $(\{4, 5\}, 5)$ -GDD of type 4^{14} which exists by Lemma 2.17. Then apply Construction 3.1 with a 4-GDD of type 3^4 to get a super-simple $(4, 5)$ -GDD of type 12^{14} .

For $u \in \{15, 18, 23\}$, there exists a $(\{4, 5\}, 1)$ -GDD of type 2^u by Lemmas 2.13 and 2.14. Applying Construction 3.1 with a super-simple $(4, 5)$ -GDD of type 6^4 or 6^5 , we get a super-simple $(4, 5)$ -GDD of type 12^u . \square

Lemma 3.19. *There exists a super-simple $(4, 5)$ -GDD of type 12^u for any $u \geq 4$.*

Proof. For $u \in [4, 12] \cup \{14, 15, 18, 19, 23\}$, the associated super-simple designs are given in Lemmas 3.6, 2.12 and 3.18. For other values of u , take a $(u, \{4, 5, 6\}, 1)$ -PBD from Lemma 3.5. Then apply Construction 3.1 with weight 12 to get a super-simple $(4, 5)$ -GDD of type 12^u . \square

Lemma 3.20. *There exists a super-simple $(4, 5)$ -GDD of type 36^u for any $u \geq 4$.*

Proof. For any $u \geq 4$, there exists a super-simple $(4, 5)$ -GDD of type 12^u by Lemma 3.19. Applying Construction 3.1 with a 4-GDD of type 3^4 , we get a super-simple $(4, 5)$ -GDD of type 36^u . \square

Theorem 3.21. *There exists a super-simple $(4, 5)$ -GDD of type g^u for $g \equiv 0 \pmod{6}$, $u \geq 4$ and $g(u-2) \geq 10$.*

Proof. Let $g = 6m$, $m \geq 1$. If $m \in \{1, 2, 6\}$, a super-simple $(4, 5)$ -GDD of type g^u comes from Lemmas 3.17, 3.19 and 3.20. If $m \neq 1, 2, 6$, there exists a super-simple $(4, 5)$ -GDD of type 6^u by Lemma 3.17. Applying Construction 3.1 with a 4-GDD of type m^4 coming from Lemma 3.5, we get a super-simple $(4, 5)$ -GDD of type g^u . \square

3.4. Case 4

We shall study the last case, that is $g \equiv 2, 4 \pmod{6}$, $u \equiv 1 \pmod{3}$ and $g(u-2) \geq 10$.

Lemma 3.22. *There exists a super-simple $(4, 5)$ -GDD of type 2^u for any $u \in \{25, 28, 31, 40, 43, 52, 55, 67\}$.*

Proof. If $u \in \{28, 40, 52\}$, there exists a super-simple $(4, 5)$ -GDD of type $(u/2)^4$ by Lemma 3.6. Applying Construction 3.2 with a super-simple $(4, 5)$ -GDD of type $2^{\frac{u}{4}}$ coming from Lemmas 2.1, 2.2 and 3.14, we get a super-simple $(4, 5)$ -GDD of type 2^u .

Next, let $u \in \{25, 31, 43, 55, 67\}$. We have a super-simple $(4, 5)$ -GDD of type 12^m for $m \in \{4, 5, 7, 9, 11\}$ by Theorem 3.21. Applying Construction 3.2 with $\eta = 1$ and a super-simple $(4, 5)$ -GDD of type 2^7 coming from Lemma 3.14, we get a super-simple $(4, 5)$ -GDD of type 2^u . \square

Lemma 3.23. *There exists a super-simple $(4, 5)$ -GDD of type 2^u for any $u \equiv 1 \pmod{3}$ and $u \geq 7$.*

Proof. Let $u = 3m + 1$, $m \geq 2$. For $m \in [2, 11] \cup \{13, 14, 17, 18, 22\}$, the associated super-simple designs are given in Lemmas 2.1–2.3, 3.14 and 3.22.

For other values of m , applying Lemma 3.7 with $\omega = 6$, we get a super-simple $(4, 5)$ -GDD of type $18^a 24^b 30^c$, where a super-simple $(4, 5)$ -GDD of type 6^h for $h \in \{4, 5, 6\}$ exists by Theorem 3.21. Then applying Construction 3.2 with $s = 2$ and $\eta = 1$, we get a super-simple $(4, 5)$ -GDD of type 2^u . \square

Lemma 3.24. *There exists a super-simple $(4, 5)$ -GDD of type 4^u for any $u \in \{13, 16\}$.*

Proof. There exists a super-simple $(u, 4, 5)$ -BIBD by Theorem 3.3. Applying Construction 3.1 with a 4-GDD of type 4^4 , we get a super-simple $(4, 5)$ -GDD of type 4^u . \square

Lemma 3.25. *There exists a super-simple $(4, 5)$ -GDD of type 4^u for any $u \in \{25, 28, 31, 40, 43, 52, 55, 67\}$.*

Proof. If $u \in \{28, 40, 52\}$, there exists a super-simple $(4, 5)$ -GDD of type u^4 by Lemma 3.6. Applying Construction 3.2 with a super-simple $(4, 5)$ -GDD of type $4^{\frac{u}{4}}$ coming from Lemma 2.6, we get a super-simple $(4, 5)$ -GDD of type 4^u .

Next, let $u \in \{25, 31, 43, 55, 67\}$. We have a super-simple $(4, 5)$ -GDD of type 24^m for $m \in \{4, 5, 7, 9, 11\}$ by Theorem 3.21. Applying Construction 3.2 with $\eta = 1$ and a super-simple $(4, 5)$ -GDD of type 4^7 coming from Lemma 2.6, we get a super-simple $(4, 5)$ -GDD of type 4^u . \square

Lemma 3.26. *There exists a super-simple $(4, 5)$ -GDD of type 4^u for any $u \equiv 1 \pmod{3}$ and $u \geq 7$.*

Proof. Let $u = 3m + 1$ and $m \geq 2$. For $m \in [2, 11] \cup \{13, 14, 17, 18, 22\}$, the associated super-simple designs are given in Lemmas 2.6, 2.7, 3.24 and 3.25.

For other values of m , applying Lemma 3.7 with $\omega = 12$, we get a super-simple $(4, 5)$ -GDD of type $36^a 48^b 60^c$. Then applying Construction 3.2 with $s = 4$ and $\eta = 1$, we get a super-simple $(4, 5)$ -GDD of type 4^u . \square

Lemma 3.27. *There exists a super-simple $(4, 5)$ -GDD of type g^u for $g \equiv 2, 4 \pmod{6}$, $g \geq 8$ and $u \equiv 1 \pmod{3}$, $u \geq 7$.*

Proof. Let $g = 6m + 2n$, $n = 1, 2$ and $m \geq 1$. There exists a super-simple $(4, 5)$ -GDD of type 2^u by Lemma 3.23. Applying Construction 3.1 with a 4-GDD of type $(3m+n)^4$ coming from Lemma 3.5, we get a super-simple $(4, 5)$ -GDD of type g^u . \square

Combining Lemmas 3.6, 3.23, 3.26 and 3.27, we have the following theorem.

Theorem 3.28. *There exists a super-simple $(4, 5)$ -GDD of type g^u for $g \equiv 2, 4 \pmod{6}$, $u \equiv 1 \pmod{3}$ and $g(u-2) \geq 10$.*

Combining Theorems 3.8, 3.13, 3.21 and 3.28, we have proved Theorem 1.2.

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